The Mathematics of Currency Hedging

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January 2011

Abstract

When should investors hold passive currency risk ? Under which conditions is there a clear gain for currency hedging ? What about prospective Sharpe ratios of both unhedged and hedged investments ? This note aims to answer those questions. It so happens that theoretical relations derived from a very simple model stick with the intuition : the gain from currency hedging depends in a non-linear way from the prospective prices of risk of the currency and the local asset, the volatility of the currency, the ratio of volatilities between currency and the local asset and the correlation between local asset and currency returns. If an investor is eager to maximize the Sharpe ratio of its foreign portfolio, such a model suggests that the case for hedging should be all the stronger as : 1) the correlation between a local asset performance and its currency return is elevated and positive 2) the local asset's prospective Sharpe ratio is large, 3) the currency's prospective Sharpe ratio is small, 4) the currency volatility is large relative to the local asset price volatility.

Foreign and Domestic Currency Returns

Consider a situation[1] where we have two currencies: the domestic currency [2] (say US dollar, USD) and the foreign emerging currency (say Brazilian Real, BRL). The spot exchange rate at time t is denoted e_t and is quoted as dollar per real:

$$e_t = \frac{\text{units of the domestic currency}}{\text{units of the foreign currency}},$$

that is $1USD = e_t \times BRL$ or $1BRL = USD/e_t$.

I assume that the domestic short rate r_d (resp. the Brazilian, or foreign short rate r_f) is deterministic and constant. We denote the corresponding risk-free money market bills B_d and B_f. I model[3] the exchange rate by a geometric Brownian motion under the physical probability $\mathbb{P}$. The drift and the volatility are both assumed constant and deterministic such that:

$$\begin{cases} de = e\left[\alpha_e dt + \sigma_e dW_e\right], \\ dB_d = r_d B_d dt, \\ dB_f = r_f B_f dt. \end{cases}$$

Let's assume that the US investor buys the foreign currency (BRL) and invests in the local risk-free rate. Such a trade is equivalent to the possibility of investing in a domestic asset with price process $\tilde{B}_f(t) = B_f(t).e_t$. Ito calculus leads to the following dynamics:

$$d\tilde{B}_f = \tilde{B}_f\left[\left(\alpha_e + r_f\right)dt + \sigma_e dW_e\right].$$

Let's introduce the currency price of risk λ_e such that:

$$d\tilde{B}_f = \tilde{B}_f\left[\left(r_d + \lambda_e\sigma_e\right)dt + \sigma_e dW_e\right], \text{ with,}$$
$$\lambda_e = \frac{\alpha_e + r_f - r_d}{\sigma_e}$$

Unhedged Fund Dynamics

In this section, the dynamics of the fund, denominated in foreign currency (BRL), is modelled as a geometric Brownian motion with constant drift and volatility:

$$dF = F\left[\left(\alpha + r_f\right)dt + \sigma dW\right].$$

[1] In this note, I will follow the approach and notations of Bjork (2003), Arbitrage Theory in Continuous Time, Oxford Chapter 17.

[2] Here I adopt the perspective from a US-based investor.

[3] Disclaimer: The views expressed in this working paper are those of the author and do not necessarily reflect those of his past and present employers.

The Brownian motion may however be correlated with the factor driving the currency dynamics.So, I denote the quadratic covariation between the two processes:

$$d\langle W, W_e\rangle = \rho dt$$

Let's assume that the investor buys the foreign currency and invest in the fund. Such a trade is equivalent to the possibility of investing in a domestic risky asset with price process $\tilde{F}(t) = F(t).e_t$. Then, after some Ito calculus, the dynamics follows:

$$\begin{aligned}
\frac{d\tilde{F}}{\tilde{F}} &= (\alpha + r_f)\, dt + \sigma dW + \alpha_e dt + \sigma_e dW_e + \rho\sigma\sigma_e dt \\
&= \frac{dF}{F} + \frac{de}{e} + \rho\sigma\sigma_e dt \\
&= \frac{dF}{F} + \frac{de}{e} + \text{ covariation drift}
\end{aligned}$$

or

$$\frac{d\tilde{F}}{\tilde{F}} = (r_f + \alpha + \alpha_e + \rho\sigma\sigma_e)\, dt + \sigma dW + \sigma_e dW_e$$

The unhedged investment in the fund is all the riskier so as the volatilities of the currency and the strategy are elevated and both risky processes are positively correlated. The expected retyrn is increasing in the foreign risk-free rate, the alpha and the currency expected return. It is increasing (resp. decreasing) in the covariation drift, if the correlation is positive (resp. negative).

Unhedged Fund Sharpe Ratio

taking expectations from the previous equation, the expected excess return for a US-based investor in the unhedged asset is:

$$\begin{aligned}
\alpha_u &= \frac{1}{dt}\mathbb{E}\left[\frac{d\tilde{F}}{\tilde{F}} - r_d dt\right] \\
&= \alpha + \rho\sigma\sigma_e + \alpha_e + r_f - r_d \\
&= \sigma\left(\lambda + \rho\sigma_e\right) + \sigma_e\lambda_e
\end{aligned}$$

Let's λ_u (resp. σ_u) denote the price of risk (or Sharpe Ratio) (resp. the volatility) of the unhedged investment:

$$\begin{aligned}
\lambda_u &= \frac{\sigma_e\lambda_e + \sigma\lambda + \rho\sigma\sigma_e}{\sigma_u}, \text{ with} \\
\sigma_u &= \left(\sigma^2 + \sigma_e^2 + 2\rho\sigma\sigma_e\right)^{\frac{1}{2}} = \sigma\left(1 + \left(\frac{\sigma_e}{\sigma}\right)^2 + 2\rho\frac{\sigma_e}{\sigma}\right)^{\frac{1}{2}}
\end{aligned}$$

Hedging Foreign Investment

Let's introduced a Hedge class $\tilde{H}$, denominated in the domestic currency (USD). Such a hedging strategy may be decomposed in three investment decisions:

1. Buying $\tilde{H}$ shares of the fund denominated in foreign currency (BRL) whose dynamics epxressed in domestic currency is:

$$\tilde{H}\frac{d\tilde{F}}{\tilde{F}} = \tilde{H}\left[\left(r_f + \alpha + \alpha_e + \rho\sigma\sigma_e\right)dt + \sigma dW + \sigma_e dW_e\right]$$

2. Borrowing short in foreign currency, whose dynamcis expressed in domestic currency is:

$$-\tilde{H}\frac{d\tilde{B}_f}{\tilde{B}_f} = -\tilde{H}\left[\left(r_f + \alpha_e\right)dt + \sigma_e dW_e\right]$$

3. Buying $\tilde{H}$ units of domestic risk-free money market bills, whose dynamics is:

$$\tilde{H}\frac{d\tilde{B}_d}{\tilde{B}_d} = \tilde{H}\left[r_d dt\right]$$

Adding those three trades together leads to the following dynamics:

$$d\tilde{H} = \tilde{H}\left[\left(r_d + \alpha + \rho\sigma\sigma_e\right)dt + \sigma dW\right]$$

The direct impact of the currency risk factor dW_e, the exchange rate drift α_e and the foreign risk-free rate r_f have been properly eliminated.

Hedged Class Return Dynamics

The return dynamics of the hedged class, expressed in domestic currency,

$$\frac{d\tilde{H}}{\tilde{H}} = \frac{dF}{F}\tilde{H} + \left(r_f - r_d\right)dt + \rho\sigma\sigma_e dt,$$

is split into the main fund return received by foregin (Brazilian) investors, augmented by the interest rate differential and supplemented by a drift term related to the covariation between the investor's currency and the fund strategy.

In real life conditions, the correlation and volatility terms are likely to be time-varying and locally stochastic. Thus a temporary rise in currency volatility may lead to a positive (resp. negative) divergence between both excess returns given the sign of the correlation term. In such a situation, the higher the correlation, the more significant a short term divergence is likely.

Hedged vs Unhegded Class Sharpe Ratio

Taking expectations from the previous equation, the alpha ($\tilde{\alpha}$) for a US-based investor in the hedged class is:

$$\begin{aligned}\tilde{\alpha} &= \frac{1}{dt}\mathbb{E}\left[\frac{d\tilde{H}}{\tilde{H}} - r_d dt\right], \\ &= \alpha + \rho\sigma\sigma_e \\ &= \sigma\left(\lambda + \rho\sigma_e\right)\end{aligned}$$

Let's λ (resp. $\tilde{\lambda}$) denote the price of risk (Sharpe ratio) of the local investment (resp. the hedged class):

$$\tilde{\lambda} = \lambda + \rho\sigma_e$$

Sharpe ratios are identical providing that the correlation between the two risk factors should be null. On the contrary, the hedged class Sharpe ratio is greater than the local investment's Sharpe ratio providing that the correlation between the currency strategy performance and the currency return is positive.

From the previous section, we may then compare the Sharpe ratios of both hedged and unhedged strategies. It follows that the price of risk of an unhedged investment is a weighted combination of the currency and hedged investment prices of risk:

$$\lambda_u = \frac{\sigma_e\lambda_e + \sigma\tilde{\lambda}}{\sigma_u}$$

Conditions required to gain from hedging (ie. that the hedged class Sharpe ratio be greater than the unhedged Sharpe ratio) follow from :

$$\begin{aligned}\tilde{\lambda} - \lambda_u &= \left(\left(\lambda + \rho\sigma_e\right)\frac{\sigma_u - \sigma}{\sigma_e} - \lambda_e\right)\frac{\sigma_e}{\sigma_u}, \\ &= \left[\left(\lambda + \rho\sigma_e\right)\frac{\sigma}{\sigma_e}\left(\left(1 + \left(\frac{\sigma_e}{\sigma}\right)^2 + 2\rho\frac{\sigma_e}{\sigma}\right)^{\frac{1}{2}} - 1\right) - \lambda_e\right]\frac{\sigma_e}{\sigma_u}\end{aligned}$$

Let's introduce the currency volatility and local (resp. unhedged strategy) asset price volatility ratios, which are by construction strictly positive:

$$\xi = \frac{\sigma_e}{\sigma}, \text{ and, } \chi = \frac{\sigma_e}{\sigma_u}$$

Then,

$$\begin{aligned}\tilde{\lambda} - \lambda_u &= \chi\left(\left(\lambda + \rho\sigma_e\right)\frac{\left(\left(1 + \xi^2 + 2\rho\xi\right)^{\frac{1}{2}} - 1\right)}{\xi} - \lambda_e\right), \\ &= \chi\left[\left(\lambda + \rho\sigma_e\right)h\left(\xi, \rho\right) - \lambda_e\right].\end{aligned}$$

As χ is strictly positive, the sign of the gain to hedge depends on the sign of the second term of the previous expression:

$$\tilde{\lambda} - \lambda_u \geq 0 \Leftrightarrow \lambda \geq \frac{\lambda_e}{h(\xi, \rho)} - \rho\sigma_e$$

We remark that h is increasing in ξ if ρ is positive. If ρ is negative, h is non-monotonous, alternatively decreasing and increasing in ξ.

To give a supplementary insight behind this non-linear relationship, let's assume that $\xi = 1$ (ie. identical volatilities)[4] and let's explore the three polar cases:

- $\rho = 1$:

$$\tilde{\lambda} - \lambda_u \geq 0 \Leftrightarrow \lambda + \sigma \geq \lambda_e$$

- $\rho = 0$:

$$\tilde{\lambda} - \lambda_u \geq 0 \Leftrightarrow \lambda \geq \frac{\lambda_e}{\sqrt{2} - 1}$$

- $\rho = -1$:

$$\tilde{\lambda} - \lambda_u \geq 0 \Leftrightarrow \sigma \geq \lambda + \lambda_e$$

We can get from this over simplified framework, that when the local asset price and the currency have a perfect positive correlation and a similar volatility, there is a stronger case for a hedged investment if the prospective Sharpe ratio of the currency is inferior to the sum of the volatility and the prospective price of risk of local asset price.

If asset and currencies exhibit a perfect negative correlation and a similar volatility, there is a stronger case for a hegded investment if the sum of the currency and asset prospective Sharpe ratios is inferior to the volatility of the asset.

[4]This assumption is not extreme as currency volatility is most of the time larger than bonds but lower than equities.

Conclusion

In the more general case, some conclusions can be drawn:

- The gain from currency hedging depends in a non-linear way from the prospective prices of risk of the currency and the local asset, the volatility of the currency, the ratio of volatilities between currency and the local asset and the correlation between local asset and currency returns.

- The lower the currency prospective price of risk, the stronger the case for hedging.

- If the correlation ρ is positive, the higher the local asset prospective price of risk or the higher the currency volatility, the stronger the case for hedging.

- If the correlation ρ is negative, there is no clear-cut conclusion. But some simulations may suggest that the closer the correlation to the zero bound, and the higher the volatility ratio (ie. the larger the currency volatility relative to the asset price volatility), the stronger the case for hedging.